Problem 1

table · soup

Intermediate Value Theorem

A pot of soup cools on a counter. Its temperature S(t), in °F, is a differentiable function of time t in minutes. Selected values are in the table. Explain why there must be a time t, 4 < t < 15, at which the temperature of the soup is 140°F.

t (minutes)	0	4	9	15	20
S(t) (°F)	190	172	150	131	124

S is differentiable and therefore continuous on $4 \le t \le 15$.

$S(15) = 131 < 140 < 172 = S(4)$

140 is between S(15) and S(4), so by the Intermediate Value Theorem there must be a time t for $4 < t < 15$ such that $S(t) = 140$.

earns the point

Pt 1: compares 140 with S(4) and S(15) from the table.

Pt 2: says continuous (from differentiable) and names the IVT.

Problem 2

formula · weather balloon

Mean Value Theorem

A weather balloon has height $h(t) = 12\sqrt{t}$ feet for $1 \le t \le 9$, with t in minutes. Show that the hypotheses of the Mean Value Theorem are met on $1 \le t \le 9$, and find the time t that the theorem guarantees.

$h(t) = 12\sqrt{t}$ is continuous for $t \ge 0$. $h'(t) = \frac{6}{\sqrt{t}}$ exists for $t > 0$.

So h is continuous on $1 \le t \le 9$ and differentiable on $1 < t < 9$.

$ARC = \frac{h(9) - h(1)}{9 - 1} = \frac{36 - 12}{8} = 3$

$\frac{6}{\sqrt{t}} = 3 \rightarrow \sqrt{t} = 2 \rightarrow t = 4$

By the Mean Value Theorem, $h'(4) = 3$ ft/min, and $1 < 4 < 9$.

earns the point

Pt 1: both hypotheses. Pt 2: ARC = 3. Pt 3: $t = 4$, inside $1 < t < 9$.

Ink color = the point it earns: Point 1 Point 2 Point 3

Problem 3

formula · ride line (calculator) — Extreme Value Theorem

The number of people in line for a ride is L(t) = 40 + 30t·e^(−t/3) for 0 ≤ t ≤ 8, with t in hours after the park opens. At what time t, 0 ≤ t ≤ 8, is the line the longest? Justify your answer.

$L'(t) = 30e^{-t/3} - 10te^{-t/3} = 10e^{-t/3}(3-t)$

$10e^{-t/3}$ is never 0, so $L'(t) = 0$ only at $t = 3$.

L is continuous on $0 \le t \le 8$, so by the Extreme Value Theorem it has an absolute maximum, at a critical point or an endpoint.

$L(0) = 40$ $\quad L(3) = 40 + 90e^{-1} \approx 73.109$

$L(8) = 40 + 240e^{-8/3} \approx 56.676$

The line is longest at $t = 3$ hours (about 73 people).

earns the point — Pt 2: candidates test, or $L' > 0$ then $L' < 0$ with only one critical point.

Problem 4

table · hiking trail (TWIST) — Mean Value Theorem

A hiker's elevation E(t), in feet, is a twice-differentiable function of time t in hours. Selected values are in the table. Explain why there must be at least one time t, 0 < t < 8, at which E′(t) = 150 feet per hour.

t (hours)	0	1	3	4	6	8
E(t) (feet)	2400	2500	2950	3150	3300	3100

$ARC = \frac{E(6) - E(0)}{6 - 0} = \frac{3300 - 2400}{6} = \frac{900}{6} = 150$

E is twice-differentiable, so E is differentiable and continuous.

By the Mean Value Theorem there must be a time t for $0 < t < 6$, which is also between 0 and 8, such that $E'(t) = 150$.

earns the point — Pt 1: $0 \le t \le 6$ gives exactly 150. Neighbors never do, and 0 to 8 gives 87.5.

Pt 2: hypotheses stated and the MVT named.

Problem 5

formula · roller coaster — Rolle's Theorem

A section of roller coaster track is modeled by $h(x) = x^3 - 6x^2 + 9x + 2$ for $0 \le x \le 3$ (h in meters, x in tens of meters). Explain why there must be a point on this section, $0 < x < 3$, where the track is level ($h'(x) = 0$). Then find every such value of x.

h is a polynomial, so h is continuous on $0 \le x \le 3$

and differentiable on $0 < x < 3$.

$h(0) = 2$ $\quad h(3) = 27 - 54 + 27 + 2 = 2$ $\quad$ so $h(0) = h(3)$

By Rolle's Theorem there is an x for $0 < x < 3$ with $h'(x) = 0$.

$h'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0 \rightarrow x = 1, x = 3$

$x = 3$ is not inside $0 < x < 3$, so the track is level at $x = 1$.

earns the point — Pt 1: continuous AND differentiable. Pt 2: $h(0) = h(3)$.

Pt 3: only $x = 1$ counts ($x = 3$ is an endpoint).

Problem 6

table + formula · two particles (TWIST, calculator) — Intermediate Value Theorem

Particle A has position $x_A(t) = 3 \sin t + t$. Particle B has position $x_B(t)$, a differentiable function; selected values are in the table. Time t is in seconds. Is there a time t, $0 < t < 4$, at which particles A and B are at the same position? Explain why or why not.

t (seconds)	0	1	2.5	4
$x_B(t)$	2	3.1	2.4	0.5

x_A is continuous, and x_B is differentiable, so x_B is continuous.

Let $D(t) = x_A(t) - x_B(t)$. D is continuous on $0 \le t \le 4$.

$D(0) = 0 - 2 = -2 < 0$

$D(4) = 3\sin 4 + 4 - 0.5 \approx 1.230 > 0$

0 is between $D(0)$ and $D(4)$, so by the Intermediate Value Theorem

there is a time t for $0 < t < 4$ with $D(t) = 0$, so $x_A(t) = x_B(t)$.

Yes, the particles are at the same position at that time.

earns the point — Pt 1: works with $x_A - x_B$ (or compares them) at $t = 0$ and $t = 4$.

Pt 2: the difference is continuous + IVT + answers yes.

Problem 7

formula · cubic (TWIST) — Extreme Value Theorem

Let f(x) = x³ − 3x² − 9x + 5. Find the absolute maximum value and the absolute minimum value of f on the closed interval −2 ≤ x ≤ 6. Justify your answers.

$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1) = 0 \rightarrow x = -1, x = 3$

f is a polynomial, so f is continuous on $-2 \le x \le 6$. By the Extreme Value Theorem f has an absolute max and an absolute min, each at a critical point or an endpoint.

$f(-2) = 3 \quad f(-1) = 10 \quad f(3) = -22 \quad f(6) = 59$

Absolute maximum value = 59 (at x = 6)

Absolute minimum value = −22 (at x = 3)

earns the point — Pt 2: all four candidates, endpoints included (a sign chart is not enough). Pt 3: both values. The max is at an endpoint.

Problem 8

piecewise · find k — Intermediate Value Theorem

f(x) = x² + 1 for x < 2, and f(x) = kx − 1 for x ≥ 2, where k is a constant. Find the value of k that makes f continuous at x = 2. For that value of k, explain why there must be a value c, 0 < c < 3, such that f(c) = 4.

From the left, $x^2 + 1 \rightarrow 5$ as $x \rightarrow 2$. $f(2) = 2k - 1$

Continuous at x = 2: $2k - 1 = 5 \rightarrow k = 3$

$f(0) = 0^2 + 1 = 1 \qquad f(3) = 3(3) - 1 = 8$

Both pieces are polynomials and f is continuous at x = 2, so f is continuous on $0 \le x \le 3$. $1 < 4 < 8$, so by the Intermediate Value Theorem there is a c for $0 < c < 3$ such that f(c) = 4.

earns the point — Pt 1: left limit = f(2), so k = 3. Pt 2: f(0) = 1 and f(3) = 8. Pt 3: continuous on the whole interval + IVT.

Problem 9

table · rowing

Rolle's Theorem

A rower's velocity v(t), in m/sec, is a differentiable function of time t in seconds. Selected values are in the table. Explain why there must be a time t, 0 < t < 12, at which the rower's acceleration is 0.

t (seconds)	0	3	5	8	12
v(t) (m/sec)	1.2	2.8	3.4	2.8	1.9

$v(3) = 2.8 = v(8)$

v is differentiable, so v is continuous on $3 \le t \le 8$.

By Rolle's Theorem there is a time t for $3 < t < 8$, which is also between 0 and 12, such that $a(t) = v'(t) = 0$.

earns the point

Pt 1: two equal velocities (or ARC = 0 on $3 \le t \le 8$).

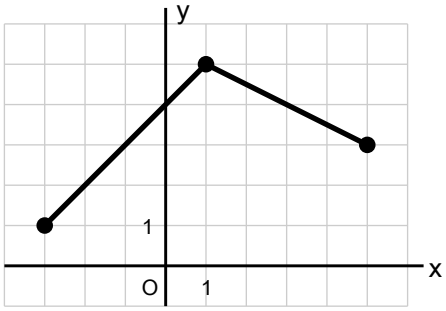
Pt 2: hypotheses + Rolle's (or the MVT) + $a(t) = v'(t) = 0$.

Ink color = the point it earns: Point 1 Point 2 Point 3

Problem 10 graph · two segments (TWIST)

Mean Value Theorem

The graph of the continuous function g, made of two line segments, is shown for $-3 \le x \le 5$. Is there a value c, $-3 < c < 5$, such that $g'(c)$ equals the average rate of change of g on $-3 \le x \le 5$? Justify your answer.



Graph of g

$ARC = \frac{g(5) - g(-3)}{5 - (-3)} = \frac{3 - 1}{8} = \frac{1}{4}$

$g'(x) = 1$ for $-3 < x < 1$ (slope of the first segment)

$g'(x) = -\frac{1}{2}$ for $1 < x < 5$ (slope of the second segment)

$g'(1)$ does not exist (corner).

No. $g'(x)$ is never $\frac{1}{4}$.

This does not contradict the MVT: g is not differentiable at $x = 1$.

earns the point — Pt 2 needs the slopes. Saying only that the MVT does not apply is not enough: no guarantee is not the same as no c.

Ink color = the point it earns: Point 1 Point 2 Point 3

Problem 11 table · how many solutions

Intermediate Value Theorem

The function f is continuous on $0 \le x \le 8$. Selected values are in the table. What is the fewest possible number of solutions to $f(x) = 2$ on $0 \le x \le 8$? Justify your answer.

x	0	2	3	5	8
f(x)	−2	1	4	3	1

f is continuous on $0 \le x \le 8$.

$f(2) = 1 < 2 < 4 = f(3)$ → by the IVT, $f(x) = 2$ for an x with $2 < x < 3$

$f(8) = 1 < 2 < 3 = f(5)$ → by the IVT, $f(x) = 2$ for an x with $5 < x < 8$

The intervals do not overlap, so $f(x) = 2$ has at least 2 solutions.

Fewest possible number of solutions = 2

earns the point — Pt 1, Pt 2: 2 is between the table values on each interval.
Pt 3: continuity + IVT + the answer 2.

Problem 12 table · highway

Mean Value Theorem

A car's distance from a toll plaza, s(t) miles, is a differentiable function of time t in hours. Selected values are in the table. The speed limit is 65 mph. Explain why the car must have been going faster than the speed limit at some time t, $0.5 < t < 0.8$.

t (hours)	0	0.25	0.5	0.8
s(t) (miles)	0	16	34	55

$$ARC = \frac{s(0.8) - s(0.5)}{0.8 - 0.5} = \frac{55 - 34}{0.3} = \frac{21}{0.3} = 70$$

s is differentiable and therefore continuous on $0.5 \le t \le 0.8$.

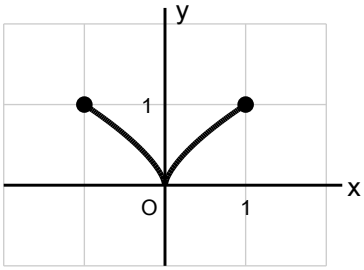
By the Mean Value Theorem there is a time t for $0.5 < t < 0.8$ such that $s'(t) = 70$ mph. $70 > 65$, so the car was speeding.

earns the point — Pt 1: average velocity = 70 on the right interval.
Pt 2: hypotheses + MVT + compares 70 with 65.

Problem 13 graph · cusp

Rolle's Theorem

Let f(x) = x^(2/3). Its graph on −1 ≤ x ≤ 1 is shown. Notice that f(−1) = f(1), but there is no value c, −1 < c < 1, with f′(c) = 0. Does this contradict Rolle's Theorem? Explain why or why not.



Graph of f(x) = x^(2/3)

$f(-1) = (-1)^{2/3} = 1 \qquad f(1) = 1$

$f'(x) = \frac{2}{3}x^{-1/3}$ is never 0, and it is undefined at $x = 0$.

No, this does not contradict Rolle's Theorem.

f is not differentiable at $x = 0$, which is inside $-1 < x < 1$,

so a hypothesis of Rolle's Theorem is not met.

earns the point — Pt 1: f′(x) found, never 0, undefined at x = 0.

Pt 2: names the failed hypothesis (not differentiable at x = 0).

Problem 14 formula · harbor tide (calculator)

Extreme Value Theorem

The depth of water at a dock is D(t) = 18 + 5 sin(πt/6) − 0.4t feet for 0 ≤ t ≤ 12, with t in hours after midnight. Find the minimum depth of the water at the dock for 0 ≤ t ≤ 12. Justify your answer.

$D'(t) = \frac{5}{6}\pi\cos\left(\frac{t}{6}\pi\right) - 0.4 = 0$

$t \approx 2.707, \quad t \approx 9.293$ (calculator)

D is continuous on $0 \le t \le 12$, so by the Extreme Value Theorem

the minimum is at a critical point or an endpoint.

$D(0) = 18 \qquad D(2.707) \approx 21.858$

$D(9.293) \approx 9.342 \qquad D(12) = 13.2$

The minimum depth is about 9.342 feet (at $t \approx 9.293$ hours).

earns the point — Pt 3: the question asks for the depth, not the time.

Ink color = the point it earns: Point 1 Point 2 Point 3